

On the stability of degenerate axially symmetric Killing horizon

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Abstract

We examine the linearized equations around extremal Kerr horizon and give some arguments towards stability of the horizon with respect to generic (non-symmetric) linear perturbation of near horizon geometry.

1 Introduction

Let us consider the following *basic equation* on a two-dimensional compact manifold

$$\omega_A||_B + \omega_B||_A + 2\omega_A\omega_B = R_{AB}, \quad (1)$$

where $\omega_A dx^A$ is a covector field, $||$ denotes covariant derivative compatible with the metric g_{AB} , and R_{AB} is its Ricci tensor. The equation (1) is a starting point of our considerations and it is a special case of (3.7) in [1], if we assume that \tilde{S}_{AB} vanishes. See also [2] or [7].

Some geometric consequences of the *basic equation*¹ (resulting from Einstein equations) were discussed in [3]. This is a non-linear PDE for unknown covector field and unknown Riemannian structure on the two-dimensional manifold. It appears in the context of Kundt's class metrics (cf. [6]), degenerate Killing horizons [2], [7], or vacuum degenerate isolated horizons [1], [8], [9]. Several important results are already proved, like topological rigidity of the horizon and integrability conditions (cf. [3]). Moreover, when the one-form $\omega_B dx^B$ is closed (e.g. static degenerate horizon [2]) there are no solutions of (1). The transformation to linear problem (invented in [3]) simplifies the proof of the uniqueness of extremal Kerr for axially symmetric horizon. However, the problem of the existence of non-symmetric solutions to the *basic equation* remains opened. In this paper we analyze a linear perturbation of extremal Kerr solution.

In [3] the following results were proved:

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¹This is an equation describing the so-called near horizon geometries see [11].

Theorem 1. *For any Riemannian metric g_{AB} on a two-dimensional, compact, connected manifold B with no boundary and genus $\mathbf{g} \geq 2$ there are no solutions of basic equation.*

Theorem 2. *For any Riemannian metric g_{AB} on a two-dimensional torus equation (1) possesses only trivial solutions $\omega^A \equiv 0 \equiv K$ and the metric g_{AB} is flat.*

Theorem 3. *There are no solutions of equation (1) with the following properties:*

- $\omega^A = 0$ only at finite set of points,
- B is a sphere with non-negative Gaussian curvature.

The symmetric part of $\omega_{A||B}$ is controlled by the equation but $f := \frac{1}{2}\omega_{A||B}\varepsilon^{AB}$ is an unknown function on a sphere. We have

$$\omega_{A||B} = f\varepsilon_{AB} + \frac{1}{2}Kg_{AB} - \omega_A\omega_B. \quad (2)$$

The integrability condition:

$$\frac{1}{4}R^{||A}{}_A + 2(R\omega^A)_{||A} = 6f^2 + \frac{3}{8}R(R - 12\omega_A\omega^A) \quad (3)$$

implies that there exists non-empty open subset, where $12\omega_A\omega^A > R > 0$.

1.1 Transformation to linear problem

Let us denote

$$\Phi_A := \frac{\omega_A}{\omega^B\omega_B}.$$

For any domain, where $\omega^B\omega_B > 0$, equation (1) implies

$$\Phi_{A||C}\varepsilon^{AC} = \left(\frac{\omega_A}{\omega^B\omega_B} \right)_{||C} \varepsilon^{AC} = 0 \quad (4)$$

which simply means that the one-form $\Phi_A dx^A$ is closed, and locally there exists a coordinate Φ such that

$$d\Phi = \Phi_A dx^A.$$

Moreover, from (1) we get

$$\Phi^A_{||A} = 1 \quad (5)$$

hence the potential Φ is a solution of the Poisson's equation:

$$\Delta\Phi = 1. \quad (6)$$

Remark If we choose one isolated point, where ω vanishes, then for a given metric g we have unique solution of the above Laplace-Beltrami equation (Green's function in the enlarged sense). For more isolated points we can take linear combination of such solutions. More precisely, let G_{x_0} be a unique solution (for a given metric g) of the equation (6) on $S^2 - \{x_0\}$. If $c_0 + c_1 + \dots + c_n = 1$ (where $c_i \in \mathbb{R}$) then $\Phi = c_0 G_{x_0} + c_1 G_{x_1} + \dots + c_n G_{x_n}$ is a solution of (6) on $S^2 - \{x_0, x_1, \dots, x_n\}$, and ω vanishes at the points x_0, x_1, \dots, x_n .

1.2 Two zeros of ω

Suppose ω_A vanishes at two distinct points in a generic way (i.e. $\omega_{A||B}$ is non-degenerate at those points). Then the equations (5) and (4) extend (in the sense of distributions) as follows:

$$\Phi^A_{||A} = 1 - c_1 \delta_{\theta=\pi} - c_2 \delta_{\theta=0} \quad (7)$$

$$\Phi_{A||C} \varepsilon^{AC} = d_1 \delta_{\theta=\pi} - d_2 \delta_{\theta=0} \quad (8)$$

Integration of the above equations on S^2 implies $d_1 = d_2 = d$ and $c_1 + c_2 = \int \lambda = (\text{total volume of } S^2)$. Hence, for $\Phi_A = \partial_A \Phi + \varepsilon_A^B \partial_B \tilde{\Phi}$ the potentials $\Phi, \tilde{\Phi}$ fulfill Laplace equations:

$$\triangle \Phi = 1 - c_1 \delta_{\theta=\pi} - c_2 \delta_{\theta=0} \quad (9)$$

$$\triangle \tilde{\Phi} = d_1 \delta_{\theta=\pi} - d_2 \delta_{\theta=0} \quad (10)$$

and their solutions may be expressed in terms of generalized Green's functions on S^2 which are well defined as the distributions (they are integrable functions, smooth outside poles with log divergence at poles).

Moreover, the trace of (1)

$$\omega^A_{||A} = K - \omega^A \omega_A \quad (11)$$

may be expressed in terms of Φ^A as follows:

$$\partial_A \left(\frac{\lambda \Phi^A}{\Phi^B \Phi_B} \right) + \frac{\lambda}{\Phi^B \Phi_B} - \lambda K = 0 \quad \equiv \quad \frac{2}{\|\Phi\|} \partial_A \left(\frac{\lambda \Phi^A}{\|\Phi\|} \right) = \lambda K. \quad (12)$$

1.3 Equivalent form of the basic equation in terms of the covector Φ_A and its conformal rescaling

Equations (4) and (5) together with (12) written as follows:

$$\lambda \varepsilon^{AC} \partial_C \Phi_A = 0, \quad (13)$$

$$\partial_A (\lambda g^{AB} \Phi_B) = \lambda, \quad (14)$$

$$\partial_A \left(\frac{\lambda g^{AB} \Phi_B}{g^{CD} \Phi_C \Phi_D} \right) + \frac{\lambda}{g^{CD} \Phi_C \Phi_D} - \lambda K = 0, \quad (15)$$

for the conformally equivalent metric $h_{AB} = \exp(-2u)g_{AB}$ (cf. eq. (36)) are almost the same

$$(\lambda K)(h) - (\lambda K)(g) = \lambda_h \triangle_h u = \lambda_g \triangle_g u, \quad -(\lambda K)(h) = \frac{1}{2} a^2_{,xx},$$

$$\partial_A (\lambda_h h^{AB} \Phi_B) = \lambda_h \exp(2u), \quad \lambda \varepsilon^{AC} \partial_C \Phi_A = 0,$$

$$\partial_A \left(\frac{\lambda_h h^{AB} \Phi_B \exp(2u)}{h^{CD} \Phi_C \Phi_D} \right) + \frac{\lambda \exp(4u)}{h^{CD} \Phi_C \Phi_D} - \lambda_h K_h + \lambda_h \triangle_h u = 0.$$

Moreover, we have the following

Theorem 4. *Equations (13–15) are locally equivalent to the eq. (2) in the domain, where $\omega_A = \frac{\Phi_A}{\Phi^B \Phi_B}$ is not vanishing.*

Proof. Let us represent tensor $\omega_{A||B}$ as a sum of three parts: skewsymmetric (f), traceless symmetric (τ_{AB}) and trace (τ):

$$\omega_{A||B} = f\varepsilon_{AB} + \tau_{AB} + \tau g_{AB}. \quad (16)$$

We have to show that τ_{AB} and τ are determined by eq. (13–15). It is easy to check that (15) implies $2\tau = K - \|\omega\|^2 = \omega^A{}_{||A}$. Moreover, (13) gives

$$\varepsilon^{AB}\omega_A\omega^C\tau_{CB} = 0$$

and similarly (14) implies

$$2\omega^A\omega^B\tau_{AB} = -\|\omega\|^4.$$

Let us observe that any two-dimensional traceless symmetric tensor has only two independent components, hence the last two conditions determine τ_{AB} uniquely in the following form:

$$\tau_{AB} = -\omega_A\omega_B + \frac{1}{2}g_{AB}\|\omega\|^2.$$

Finally, the above formula together with $\tau = \frac{1}{2}(K - \|\omega\|^2)$ give the eq. (2). \square

One can also check the following formula:

$$\Phi_{A||B} = 0 \cdot \varepsilon_{AB} + \frac{1}{2}g_{AB} - f(*\Phi_A\Phi_B + *\Phi_B\Phi_A) + (1 - K\|\Phi\|^2) \left(\frac{\Phi_A\Phi_B}{\|\Phi\|^2} - \frac{1}{2}g_{AB} \right) \quad (17)$$

which is equivalent to (2) but in terms of Φ .

Let us observe that $\Phi^B{}_{||BA} = 0$ hence the symmetry of the tensor $\Phi_{A||B}$ implies

$$\Phi_A{}^{||B}{}_B = \Phi^B{}_{||AB} = \Phi^B{}_{||AB} - \Phi^B{}_{||BA} = R_{AB}\Phi^B,$$

and we obtain the following nice formulae:

$$\Phi^{A||B}{}_B = K\Phi^A, \quad *\Phi^{A||B}{}_B = K*\Phi^A. \quad (18)$$

Moreover,

$$\Phi^{A||B}\omega_{A||B} = K - \|\omega\|^2$$

and

$$(\Phi^{A||B}\omega_A)_{||B} = \Phi_A{}^{||B}{}_B\omega^A + \Phi^{A||B}\omega_{A||B} = 2K - \|\omega\|^2,$$

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial S_\epsilon} \Phi^{A||B}\omega_A dS_B = \int_{S^2} K = 4\pi \quad \text{where } S_\epsilon := S^2 \setminus \left(\bigcup_{x_i \in \omega^{-1}(\{0\})} K(x_i, \epsilon) \right).$$

1.4 Solution of the problem with axial symmetry

Let us consider axially symmetric two-metric on a sphere in the following form:

$$g = 2m^2 [A^{-1}(\theta)d\theta^2 + A(\theta)\sin^2\theta d\phi^2], \quad (19)$$

where $A : [0, \pi] \rightarrow \mathbb{R}$ is a positive smooth function with boundary values $A(0) = A(\pi) = 1$, and positive constant m^2 controls the total volume of a sphere. Eq. (19) implies that $\lambda = \sqrt{\det g_{AB}} = 2m^2 \sin \theta$. From (5) we get

$$\partial_A (\lambda \Phi^A) = \lambda,$$

hence

$$\lambda \Phi^\theta = -2m^2(\cos \theta + C),$$

where C is a constant of integration. Moreover, from (4) we obtain $\partial_\theta \Phi_\phi = 0$ and

$$\Phi_\phi = 2m^2 \alpha$$

with arbitrary constant α . For the trace of (1) we get

$$\omega^A_{||A} = K - \omega^A \omega_A, \quad (20)$$

which, in terms of Φ^A , takes the following form:

$$\partial_A \left(\frac{\lambda \Phi^A}{\Phi^B \Phi_B} \right) + \frac{\lambda}{\Phi^B \Phi_B} - \lambda K = 0. \quad (21)$$

The square of vector Φ^A :

$$\Phi^A \Phi_A = 2m^2 \frac{(\cos \theta + C)^2 + \alpha^2}{A \sin^2 \theta},$$

Gaussian curvature:

$$\lambda K = -\frac{1}{2} \partial_\theta \left[\frac{1}{\sin \theta} \partial_\theta (A \sin^2 \theta) \right]$$

and the eq. (21) imply that the function A obeys the following linear ODE:

$$\frac{d}{dx} \frac{(x+C)y}{(x+C)^2 + \alpha^2} + \frac{y}{(x+C)^2 + \alpha^2} + \frac{1}{2} \frac{d^2 y}{dx^2} = 0, \quad (22)$$

where $x := \cos \theta$ and $y := A \sin^2 \theta$. For $\alpha = 0$ we get

$$\frac{d^2}{dx^2} [(x+C)y] = 0$$

with a general solution $y = \frac{ax+b}{x+C}$. However, in the case $\alpha = 0$ the function $A = \frac{y}{1-x^2}$ can not be regular at both points $+1$ and -1 simultaneously. Nonexistence of regular

solutions for $\alpha = 0$ confirms the main result of [2], because $\Phi_\phi = 0$ gives $\omega_\phi = 0$ which obviously implies $d\omega = 0$.

For $\alpha \neq 0$ we take a new variable $t := \frac{x+C}{\alpha}$, and the equation (22) takes the form

$$\frac{d}{dt} \left[\frac{d}{dt}(ty) - \frac{2y}{1+t^2} \right] = 0$$

with the following general solution

$$y = \frac{at + b(t^2 - 1)}{t^2 + 1} \quad (23)$$

with arbitrary constants a, b . The solution (23) gives the following form of the function A :

$$A = \frac{y}{1-x^2} = \frac{a\alpha(x+C) + b[(x+C)^2 - \alpha^2]}{(1-x^2)[(x+C)^2 + \alpha^2]}.$$

Regularity of A at $x = \pm 1$ implies that $C^2 = 1 - \alpha^2$ (hence $0 < |\alpha| \leq 1$) and $\frac{a}{b}\alpha + 2C = 0$ which gives

$$A = \frac{-b}{[(x+C)^2 + \alpha^2]}.$$

Moreover, $A(\pm 1) = 1$ implies $b = -2$, $\alpha = 1$, $C = 0$, hence

$$A = \frac{2}{1+x^2} = \frac{2}{1+\cos^2 \theta},$$

and finally

$$g_{\text{Kerr}} = 2m^2 \left[\frac{1 + \cos^2 \theta}{2} d\theta^2 + \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} d\phi^2 \right] \quad (24)$$

and

$$\omega^\theta = -\frac{\sin \theta \cos \theta}{m^2(1 + \cos^2 \theta)^2}, \quad \omega^\varphi = \frac{1}{2m^2(1 + \cos^2 \theta)}, \quad (25)$$

which corresponds to extremal Kerr with mass m and angular momentum m^2 .

It is worth to notice that the solution (25) in terms of Φ_A has a simple and natural form. More precisely, equations (4) and (5) extended through the “poles” are the following:

$$\Phi_{A||C} \varepsilon^{AC} = 4\pi m^2 (\delta_{\theta=\pi} - \delta_{\theta=0}), \quad (26)$$

$$\Phi_A^A|_A = 1 - 4\pi m^2 (\delta_{\theta=\pi} + \delta_{\theta=0}), \quad (27)$$

where by δ_p we denote a Dirac delta at point p , and $8\pi m^2 (= \int \lambda)$ is a total volume of the sphere (24).

Let G_p be a Green's function satisfying

$$\begin{cases} \Delta G_p = 1 - 8\pi m^2 \delta_p, \\ \int \lambda G_p = 0. \end{cases} \quad (28)$$

The potentials $\Phi, \tilde{\Phi}$ for the covector field Φ_A defined (up to a constant) as follows:

$$\Phi_A = \partial_A \Phi + \varepsilon_A{}^B \partial_B \tilde{\Phi} \quad (29)$$

take a simple form

$$\Phi = \frac{1}{2}(G_{\theta=0} + G_{\theta=\pi}), \quad (30)$$

$$\tilde{\Phi} = \frac{1}{2}(G_{\theta=0} - G_{\theta=\pi}), \quad (31)$$

because equations (26), (27) and (29) imply

$$\Delta \Phi = 1 - 4\pi m^2 (\delta_{\theta=\pi} + \delta_{\theta=0}), \quad (32)$$

$$\Delta \tilde{\Phi} = 4\pi m^2 (\delta_{\theta=\pi} - \delta_{\theta=0}). \quad (33)$$

Moreover, the Green's functions for extremal Kerr (24) are given in the explicit form:

$$G_{\theta=0} = 4m^2 \left[\frac{1}{2} \sin^2 \frac{\theta}{2} + \frac{1}{8} \sin^2 \theta - \log(\sin \frac{\theta}{2}) + \frac{1}{3} \right], \quad (34)$$

$$G_{\theta=\pi} = 4m^2 \left[\frac{1}{2} \cos^2 \frac{\theta}{2} + \frac{1}{8} \sin^2 \theta - \log(\cos \frac{\theta}{2}) + \frac{1}{3} \right]. \quad (35)$$

2 Linearization of basic equation around extremal Kerr

After introducing a new coordinate $x := \cos \theta$ the (two-dimensional) extremal Kerr (24–25) takes the following form:

$$g_{\text{Kerr}} = h_{AB} dx^A dx^B = 2m^2 (a^{-2} dx^2 + a^2 d\varphi^2), \quad (36)$$

where $a^2 := 2 \frac{1-x^2}{1+x^2}$ and $\lambda := \sqrt{\det h_{AB}} = 2m^2$. The components of various objects for Kerr are the following:

$$\begin{aligned} \omega_x &= \frac{x}{1+x^2}, \quad \omega_\varphi = \frac{a^2}{1+x^2}, \quad \|\omega\|^2 = \frac{1}{2m^2} \frac{a^2}{1+x^2}. \\ \frac{1}{2m^2} \Phi &= \frac{x}{a^2} dx + d\varphi, \quad \frac{1}{2m^2} * \Phi = \frac{1}{a^2} dx - x d\varphi \quad (*\Phi_A := \varepsilon_A{}^B \Phi_B), \end{aligned} \quad (37)$$

$$K = \frac{2}{m^2} \frac{1-3x^2}{(1+x^2)^3}, \quad f = \frac{1}{m^2} \frac{x(1+a^2)}{(1+x^2)^2} = \frac{1}{m^2} \frac{x(3-x^2)}{(1+x^2)^3}, \quad \frac{K}{2} + if = \frac{1}{m^2(1-ix)^3}, \quad (38)$$

$$\|\Phi\| = \|\ast\Phi\|, \quad \ast\Phi \wedge \Phi = \|\Phi\|^2 \lambda dx \wedge d\varphi. \quad (39)$$

The nearby metric g we describe by conformal factor:

$$g_{AB} = \exp(2u)h_{AB} \quad (40)$$

and we get

$$\begin{aligned} \Gamma^C_{AB}(g) &= \Gamma^C_{AB}(h) + S^C_{AB}, \\ S^C_{AB} &= \delta^C_A \partial_B u + \delta^C_B \partial_A u - h_{AB} h^{CD} \partial_D u. \end{aligned}$$

Let us denote by $u^B := h^{BA} \partial_A u$ the gradient of u with respect to the metric h . We have

$$\begin{aligned} \nabla_B(g)\omega_A &= \nabla_B(h)\omega_A - S^C_{AB}(u)\omega_C \\ &= \nabla_B(h)\omega_A + h_{AB}\omega_C u^C - \omega_A u_B - \omega_B u_A \end{aligned} \quad (41)$$

Moreover, the Gaussian curvatures K_h and K_g for the conformally related metrics h and g respectively are related as follows

$$\triangle_h u = K_h - \exp(2u)K_g.$$

This gives the following transformation for the right-hand side of (1):

$$R_{AB}(g) = K_g g_{AB} = (K_h - \triangle_h u) h_{AB}. \quad (43)$$

Using (41) and (43) we rewrite basic equation (1) as follows:

$$\nabla_B(h)\omega_A + \nabla_A(h)\omega_B + 2(h_{AB}\omega_C u^C - \omega_A u_B - \omega_B u_A + \omega_A \omega_B) = (K_h - \triangle_h u) h_{AB}. \quad (44)$$

Let us denote the linear part of the covector ω by

$$\mathbf{w}_A := \omega_A - \omega_A^{\text{Kerr}}.$$

Now we are ready to linearize basic equation.

$$\begin{aligned} 2(\omega_A^{\text{Kerr}} \mathbf{w}_B + \omega_B^{\text{Kerr}} \mathbf{w}_A + h_{AB} \omega_C^{\text{Kerr}} u^C - \omega_A^{\text{Kerr}} u_B - \omega_B^{\text{Kerr}} u_A) + \nabla_B(h) \mathbf{w}_A + \nabla_A(h) \mathbf{w}_B \\ + h_{AB} \triangle_h u = 2\mathbf{w}_A u_B + 2\mathbf{w}_B u_A - 2h_{AB} \mathbf{w}_C u^C - 2\mathbf{w}_A \mathbf{w}_B \approx 0 \end{aligned} \quad (45)$$

Finally, for covector \mathbf{w}_A and conformal factor u in (40) the linearization of (1) takes the following form:

$$\nabla_A(\mathbf{w}^A + u^A) + 2\omega^A \mathbf{w}_A = 0, \quad (46)$$

$$TS(\nabla_A \mathbf{w}_B + 2\omega_A(\mathbf{w}_B - u_B)) = 0, \quad (47)$$

where now ω and ∇ are background objects (corresponding to the Kerr solution (36), and

$$TS(t_{AB}) := t_{AB} + t_{BA} - h_{AB}h^{CD}t_{CD}$$

denotes the traceless symmetric part of the tensor t_{AB} .

We show in Appendix B that after elimination of u_A we get:

$$\Delta_h(\mathbf{w}_A * \Phi^A) + \varepsilon^{AB} \mathbf{w}_{A||B} = 0, \quad (48)$$

$$\Delta_h(\mathbf{w}_A \Phi^A) + 4\mathbf{w}_A \Phi^A \|\omega\|^2 + 3\mathbf{w}^A_{||A} = 0, \quad (49)$$

where

$$u_A = \frac{1}{2} [\mathbf{w}_A + \nabla_B(\Phi^B \mathbf{w}_A - \Phi_A \mathbf{w}^B) + \nabla_A(\Phi^B \mathbf{w}_B)] . \quad (50)$$

Remark: The equations (48–49) are *conformally* covariant with respect to the rescaling of the two-metric h . More precisely, the form of these equations is the same for two conformally related metrics provided that $\Phi, * \Phi$ are vector fields and \mathbf{w} and ω are covector fields. One can easily verify this observation multiplying the above equations by scalar density λ .

The non-existence of the solution \mathbf{w}_A to the equations (48–49) is equivalent to the stability of the solution (24–25).

Axial symmetry of the background solution enables one to separate variable φ with the help of Fourier transform and (48–49) becomes second order ODE for

$$\mathbf{w} : [-1, 1] \mapsto \mathbb{R}^2 .$$

One can also introduce another pair of variables:

$$\alpha := \varepsilon^{AB} \mathbf{w}_A \Phi_B = \frac{1}{2} \left[2\mathbf{w}_x - \frac{x(1+x^2)}{1-x^2} \mathbf{w}_\varphi \right] = \mathbf{w}_x - \frac{x(1+x^2)}{2(1-x^2)} \mathbf{w}_\varphi = \mathbf{w}_x - \frac{x}{a^2} \mathbf{w}_\varphi ,$$

$$\beta := \Phi_A \mathbf{w}^A = m^2 \left[\frac{x(1+x^2)}{1-x^2} \mathbf{w}^x + 2\mathbf{w}^\varphi \right] = x\mathbf{w}_x + \frac{1+x^2}{2(1-x^2)} \mathbf{w}_\varphi = x\mathbf{w}_x + \frac{1}{a^2} \mathbf{w}_\varphi ,$$

where $a^2 := 2 \frac{1-x^2}{1+x^2}$. The formula (50) takes a simple form:

$$u_A = \frac{1}{2} [\mathbf{w}_A + \varepsilon_A^B \nabla_B \alpha + \nabla_A \beta] . \quad (51)$$

Moreover, the inverse transformation

$$\mathbf{w}_x = \frac{\alpha + x\beta}{1+x^2}, \quad \mathbf{w}_\varphi = a^2 \frac{\beta - x\alpha}{1+x^2}, \quad (52)$$

implies the following form of the equations (48–49) in terms of variables α, β :

$$\Delta_h(\alpha) + \partial_\varphi \left(\frac{\alpha + x\beta}{1+x^2} \right) - \partial_x \left(a^2 \frac{\beta - x\alpha}{1+x^2} \right) = 0, \quad (53)$$

$$\Delta_h(\beta) + \frac{4a^2}{1+x^2}\beta + 3\partial_\varphi\left(\frac{\beta-x\alpha}{1+x^2}\right) + 3\partial_x\left(a^2\frac{\alpha+x\beta}{1+x^2}\right) = 0, \quad (54)$$

where

$$\Delta_h := \partial_x a^2 \partial_x + \partial_\varphi a^{-2} \partial_\varphi.$$

Let us denote $v := \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $B := \frac{1}{1+x^2} \begin{bmatrix} x & -1 \\ 3 & 3x \end{bmatrix}$, $C := \frac{1}{1+x^2} \begin{bmatrix} 1 & x \\ -3x & 3 \end{bmatrix}$, then the equations (53–54) take the following (matrix) form:

$$\Delta_h v + \frac{4a^2}{1+x^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v + \partial_x (a^2 B v) + \partial_\varphi (C v) = 0. \quad (55)$$

Other useful identities:

$$\mathbf{w}_A = \beta \omega_A + \alpha \varepsilon_A^B \omega_B = \partial_A (2u - \beta) - \varepsilon_A^B \partial_B \alpha,$$

$$2\partial_A u = \partial_A \beta + \beta \omega_A + \varepsilon_A^B (\partial_B \alpha + \alpha \omega_B).$$

2.1 Boundary data

A small perturbation of Kerr data (24–25) does not destroy the number of two zeros for covector ω_A . This is a simple consequence of the “inverse function theorem”. More precisely, the non-vanishing curvature in the neighborhood of “spherical pole” (zero of ω_A) assures invertibility of the first derivative $\nabla_A \omega_B$ in a small open neighborhood² and implies existence of a local diffeomorphism $\omega_A(x^B)$. Hence, for perturbed $\omega_A(x^B)$ there exists (in a small open neighborhood of spherical pole) precisely one point, where ω_A vanishes. The freedom of global conformal transformations enables one to introduce “new conformal coordinates” in such a way that the spherical poles are always at the points where ω_A vanishes. Hence, we can always assume that the perturbed ω_A vanishes at spherical poles which implies zero (homogeneous) boundary data for linear perturbation \mathbf{w}_A or equivalently for $v = (\alpha, \beta)$ ³.

One can also show that respectively chosen conformal vector field X enables one to change $\mathbf{w}_A \rightarrow \mathbf{w}_A + \mathcal{L}_X \omega_A$ in such a way that it will vanish at a given point (see appendix D).

Theorem 5. *The equation (55) has no regular solutions for homogeneous boundary data $\mathbf{w}_A|_{x=1} = 0 = \mathbf{w}_A|_{x=-1}$.*

The above Theorem implies stability of the extremal Kerr horizon.

Proof. Let us consider Fourier series for v :

$$v(x, \phi) = \sum_{k=-\infty}^{\infty} v_k(x) e^{ik\phi}$$

²Formula (2) implies that $\det \nabla_A \omega^B = f^2 + \frac{K}{2}(\frac{K}{2} - \|\omega\|^2)$, for Kerr $\det \nabla_A \omega^B = \frac{2x^2(3-x^2)}{(1+x^2)^5}$ and it vanishes only on the equator $x = 0$.

³It is not obvious that $\mathbf{w}_A = 0$ corresponds to $v = 0$ and it is not true for $k = 0$.

It leads to ODE for $v_k(x)$:

$$\partial_x a^2 \partial_x v_k - \frac{k^2}{a^2} v_k + \frac{4a^2}{1+x^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v_k + \partial_x(a^2 B v_k) + ik(C v_k) = 0 \quad (56)$$

We check that v_k vanishes at poles for $|k| \geq 1$, because \mathbf{w}_A vanishes there. For $k = 0$ we have axial symmetry, hence we already have uniqueness in full nonlinear case, however it would be nice to check this fact independently.

For $|k| > 8$ we prove in appendix C that there are no regular solutions. There are some initial numerical results which confirm nonexistence hypothesis for $|k| \leq 8$. We are going to check numerically the existence or nonexistence of low modes. The results will be published in a separate article. \square

A Kerr in conformal coordinates

The background metric (36) can be conformally related to unit sphere metric as follows:

$$g_{\text{Kerr}} = h_{AB} dx^A dx^B = 2m^2 (a^{-2} dx^2 + a^2 d\varphi^2) = 2m^2 F^2 \tilde{h}_{AB} d\tilde{x}^A d\tilde{x}^B \quad (57)$$

where

$$\tilde{h}_{AB} d\tilde{x}^A d\tilde{x}^B := \left[\frac{d\tilde{x}^2}{1-\tilde{x}^2} + (1-\tilde{x}^2) d\tilde{\varphi}^2 \right], \quad \tilde{\varphi} = \varphi, \quad \tilde{x} = \frac{x - \tanh \frac{x}{2}}{1 - x \tanh \frac{x}{2}}$$

is the usual unit sphere metric and

$$F^2 = \frac{a^2}{1-\tilde{x}^2} = \frac{2}{1+x^2} \left(\cosh \frac{x}{2} - x \sinh \frac{x}{2} \right)^2, \quad dx = F^2 d\tilde{x}.$$

B Reduced linearized equations

B.1 Elimination of u_A

We start from traceless part (47):

$$\begin{aligned} & \nabla_A \mathbf{w}_B + \nabla_B \mathbf{w}_A + 2\omega_A \mathbf{w}_B + 2\omega_B \mathbf{w}_A - \nabla^C \mathbf{w}_C h_{AB} - 2\omega^C \mathbf{w}_C h_{AB} + \\ & - 2\omega_A u_B - 2\omega_B u_A + 2\omega^C u_C h_{AB} = 0. \end{aligned} \quad (58)$$

The two independent components $(AB) = (xx)$ and $(AB) = (x\phi)$ can be written as follows. Component (xx) :

$$\begin{aligned} & 2\nabla_x \mathbf{w}_x + 4\omega_x \mathbf{w}_x - (\nabla^x \mathbf{w}_x + \nabla^\phi \mathbf{w}_\phi) h_{xx} - 2(\omega^x \mathbf{w}_x + \omega^\phi \mathbf{w}_\phi) h_{xx} + \\ & - 4\omega_x u_x + 2(\omega^x u_x + \omega^\phi u_\phi) h_{xx} = 0 \end{aligned}$$

or in an equivalent form (dividing by h_{xx}):

$$2\nabla^x \mathbf{w}_x + 4\omega^x \mathbf{w}_x - \nabla^x \mathbf{w}_x - \nabla^\phi \mathbf{w}_\phi - 2\omega^x \mathbf{w}_x - 2\omega^\phi \mathbf{w}_\phi - 4\omega^x u_x + 2\omega^x u_x + 2\omega^\phi u_\phi = 0,$$

$$2\omega^x u_x - 2\omega^\phi u_\phi = \nabla^x \mathbf{w}_x - \nabla^\phi \mathbf{w}_\phi + 2\omega^x \mathbf{w}_x - 2\omega^\phi \mathbf{w}_\phi. \quad (59)$$

Component $(x\phi)$:

$$\nabla_x \mathbf{w}_\phi + \nabla_\phi \mathbf{w}_x + 2\omega_x \mathbf{w}_\phi + 2\omega_\phi \mathbf{w}_x - 2\omega_x u_\phi - 2\omega_\phi u_x = 0,$$

$$2\omega_\phi u_x + 2\omega_x u_\phi = \nabla_x \mathbf{w}_\phi + \nabla_\phi \mathbf{w}_x + 2\omega_x \mathbf{w}_\phi + 2\omega_\phi \mathbf{w}_x. \quad (60)$$

Finally we have (in matrix form)

$$\begin{bmatrix} 2\omega^x & -2\omega^\phi \\ 2\omega_\phi & 2\omega_x \end{bmatrix} \begin{bmatrix} u_x \\ u_\phi \end{bmatrix} = \begin{bmatrix} \nabla^x \mathbf{w}_x - \nabla^\phi \mathbf{w}_\phi + 2\omega^x \mathbf{w}_x - 2\omega^\phi \mathbf{w}_\phi \\ \nabla_x \mathbf{w}_\phi + \nabla_\phi \mathbf{w}_x + 2\omega_x \mathbf{w}_\phi + 2\omega_\phi \mathbf{w}_x \end{bmatrix}. \quad (61)$$

Let us denote $A := \begin{bmatrix} 2\omega^x & -2\omega^\phi \\ 2\omega_\phi & 2\omega_x \end{bmatrix}$. Hence

$$A^{-1} = \frac{1}{4(\omega^x \omega_x + \omega^\phi \omega_\phi)} \begin{bmatrix} 2\omega_x & 2\omega^\phi \\ -2\omega_\phi & 2\omega^x \end{bmatrix} = \frac{1}{2\|\omega\|^2} \begin{bmatrix} \omega_x & \omega^\phi \\ -\omega_\phi & \omega^x \end{bmatrix}.$$

Multiplying by A^{-1} we get

$$u_x = \frac{1}{2\|\omega\|^2} (\omega_x \nabla^x \mathbf{w}_x - \omega_x \nabla^\phi \mathbf{w}_\phi + 2\omega_x \omega^x \mathbf{w}_x - 2\omega_x \omega^\phi \mathbf{w}_\phi + \omega^\phi \nabla_x \mathbf{w}_\phi + \omega^\phi \nabla_\phi \mathbf{w}_x + 2\omega_x \omega^\phi \mathbf{w}_\phi + 2\omega_\phi \omega^\phi \mathbf{w}_x)$$

or in simpler form

$$u_x = \frac{1}{2\|\omega\|^2} (\omega^x \nabla_x \mathbf{w}_x - \omega_x \nabla^\phi \mathbf{w}_\phi + \omega^\phi \nabla_x \mathbf{w}_\phi + \omega^\phi \nabla_\phi \mathbf{w}_x + 2\|\omega\|^2 \mathbf{w}_x). \quad (62)$$

Similarly, component ϕ :

$$u_\phi = \frac{1}{2\|\omega\|^2} (\omega_\phi \nabla^\phi \mathbf{w}_\phi - \omega_\phi \nabla^x \mathbf{w}_x + 2\omega_\phi \omega^\phi \mathbf{w}_\phi - 2\omega_\phi \omega^x \mathbf{w}_x + \omega^x \nabla_x \mathbf{w}_\phi + \omega^x \nabla_\phi \mathbf{w}_x + 2\omega_x \omega^\phi \mathbf{w}_\phi + 2\omega^x \omega_\phi \mathbf{w}_x)$$

or

$$u_\phi = \frac{1}{2\|\omega\|^2} (\omega_\phi \nabla^\phi \mathbf{w}_\phi - \omega_\phi \nabla^x \mathbf{w}_x + \omega^x \nabla_x \mathbf{w}_\phi + \omega^x \nabla_\phi \mathbf{w}_x + 2\|\omega\|^2 \mathbf{w}_\phi). \quad (63)$$

Equations (62) and (63) we can rewrite in covariant form:

$$u_A = \frac{1}{2\|\omega\|^2} (\omega^B \nabla_B \mathbf{w}_A + \omega^B \nabla_A \mathbf{w}_B - \omega_A \nabla^B \mathbf{w}_B + 2\|\omega\|^2 \mathbf{w}_A)$$

Now, introducing $\Phi_A := \frac{1}{\|\omega\|^2} \omega_A$ we have

$$u_A = \mathbf{w}_A + \frac{1}{2} (\Phi^B \nabla_B \mathbf{w}_A + \Phi^B \nabla_A \mathbf{w}_B - \Phi_A \nabla^B \mathbf{w}_B). \quad (64)$$

Let us notice the following

$$\Phi^B \nabla_B \mathbf{w}_A = \nabla_B (\Phi^B \mathbf{w}_A) - \mathbf{w}_A \nabla_B \Phi^B = \nabla_B (\Phi^B \mathbf{w}_A) - \mathbf{w}_A \quad (65)$$

(from [3] we know that $\nabla_B \Phi^B = 1$),

$$\Phi^B \nabla_A \mathbf{w}_B = \nabla_A (\Phi^B \mathbf{w}_B) - \mathbf{w}_B \nabla_A \Phi^B = \nabla_A (\Phi^B \mathbf{w}_B) - \mathbf{w}^B \nabla_A \Phi_B, \quad (66)$$

$$\Phi_A \nabla^B \mathbf{w}_B = \nabla^B (\Phi_A \mathbf{w}_B) - \mathbf{w}_B \nabla^B \Phi_A = \nabla_B (\Phi_A \mathbf{w}^B) - \mathbf{w}^B \nabla_B \Phi_A. \quad (67)$$

The above equations (64), (65), (66) and (67) imply

$$u_A = \mathbf{w}_A + \frac{1}{2} [\nabla_B (\Phi^B \mathbf{w}_A - \Phi_A \mathbf{w}^B) + \nabla_A (\Phi^B \mathbf{w}_B) + \mathbf{w}^B (\nabla_B \Phi_A - \nabla_A \Phi_B) - \mathbf{w}_A].$$

From [3] we know that $\varepsilon^{AB} \nabla_B \Phi_A = 0$, hence $\nabla_B \Phi_A - \nabla_A \Phi_B = 0$, and we obtain formula (50): $u_A = \frac{1}{2} [\mathbf{w}_A + \nabla_A (\Phi^B \mathbf{w}_B) + \nabla_B (\Phi^B \mathbf{w}_A - \Phi_A \mathbf{w}^B)]$.

B.2 Equations for \mathbf{w}_A

The trace and curl of u_A gives:

$$\nabla^A \mathbf{w}_A + 2\omega^A \mathbf{w}_A + \nabla^A u_A = 0, \quad (68)$$

$$\varepsilon^{AB} \nabla_B u_A = 0 \quad (69)$$

Using formula

$$u_A = \frac{1}{2} [\mathbf{w}_A + \nabla_A (\Phi^B \mathbf{w}_B) + \nabla_B (\Phi^B \mathbf{w}_A - \Phi_A \mathbf{w}^B)] \quad (70)$$

and equation (68) we obtain

$$\nabla^A \mathbf{w}_A + 2\omega^A \mathbf{w}_A + \frac{1}{2} \nabla^A \mathbf{w}_A + \frac{1}{2} \nabla^A \nabla_A (\Phi^B \mathbf{w}_B) + \frac{1}{2} \nabla^A \nabla_B (\Phi^B \mathbf{w}_A - \Phi_A \mathbf{w}^B) = 0,$$

$$\Delta (\Phi^B \mathbf{w}_B) + 3\nabla^A \mathbf{w}_A + 4\omega^A \mathbf{w}_A + \nabla^A \nabla^B (\Phi_B \mathbf{w}_A - \Phi_A \mathbf{w}_B) = 0.$$

Moreover, $\nabla^A \nabla^B (\Phi_B \mathbf{w}_A - \Phi_A \mathbf{w}_B) = 0$, because

$$\nabla_C \nabla_D t^{AB} - \nabla_D \nabla_C t^{AB} = R^A_{ECD} t^{EB} + R^B_{ECD} t^{AE},$$

where by R^A_{BCD} we denote Riemann curvature tensor. We have

$$\nabla_A \nabla_B t^{AB} - \nabla_B \nabla_A t^{AB} = R_{EB} t^{EB} - R_{EA} t^{AE},$$

where by R_{AB} we denote Ricci tensor. The symmetry of Ricci

$$\nabla_A \nabla_B t^{AB} - \nabla_B \nabla_A t^{AB} = R_{EB} t^{EB} - R_{AE} t^{AE} = 0$$

implies

$$\begin{aligned}\nabla^A \nabla^B (\Phi_B \mathbf{w}_A - \Phi_A \mathbf{w}_B) &= \nabla^B \nabla^A (\Phi_B \mathbf{w}_A - \Phi_A \mathbf{w}_B) = \\ &= \nabla^A \nabla^B (\Phi_A \mathbf{w}_B - \Phi_B \mathbf{w}_A) = -\nabla^A \nabla^B (\Phi_B \mathbf{w}_A - \Phi_A \mathbf{w}_B).\end{aligned}$$

Hence $\nabla^A \nabla^B (\Phi_B \mathbf{w}_A - \Phi_A \mathbf{w}_B) = 0$ and we obtain (49)

$$\Delta(\Phi^B \mathbf{w}_B) + 3\nabla^A \mathbf{w}_A + 4\omega^A \mathbf{w}_A = 0.$$

Using formula (70) and equation (69) we get

$$\varepsilon^{AB} \nabla_B \mathbf{w}_A + \varepsilon^{AB} \nabla_B \nabla^C (\Phi_C \mathbf{w}_A - \Phi_A \mathbf{w}_C) + \varepsilon^{AB} \nabla_B \nabla_A (\Phi^B \mathbf{w}_B) = 0.$$

Vanishing torsion gives $\varepsilon^{AB} \nabla_B \nabla_A (\Phi^B \mathbf{w}_B) = 0$, hence

$$\varepsilon^{AB} \nabla_B \mathbf{w}_A + \varepsilon^{AB} \nabla_B \nabla^C (\Phi_C \mathbf{w}_A - \Phi_A \mathbf{w}_C) = 0.$$

Moreover, $\Phi_C \mathbf{w}_A - \Phi_A \mathbf{w}_C = \varepsilon_{AC} \varepsilon^{DE} \Phi_D \mathbf{w}_E$ implies

$$\varepsilon^{AB} \nabla_B \mathbf{w}_A + \varepsilon^{AB} \varepsilon_{AC} \nabla_B \nabla^C (\varepsilon^{DE} \Phi_D \mathbf{w}_E) = 0.$$

Using identity $\varepsilon^{AB} \varepsilon_{AC} = -\delta^B_C$, we get

$$\varepsilon^{AB} \nabla_B \mathbf{w}_A + \nabla_C \nabla^C (\varepsilon^{AB} \Phi_B \mathbf{w}_A) = 0,$$

and finally we obtain (48)

$$\Delta(\varepsilon^{AB} \Phi_B \mathbf{w}_A) + \varepsilon^{AB} \nabla_B \mathbf{w}_A = 0.$$

C Proof for large k

Stability for the extremal Kerr leads to the following equation:

$$\partial_x a^2 \partial_x - \frac{k^2}{a^2} v_k + D v_k + \partial_x (a^2 B v_k) + i k C v_k = 0, \quad (71)$$

where

- $a^2 = 2 \frac{1-x^2}{1+x^2}$
- $v_k : [-1, 1] \rightarrow \mathbb{C}^2$ is the unknown function we are looking for,
- $B = \frac{1}{1+x^2} \begin{bmatrix} x & -1 \\ 3 & 3x \end{bmatrix},$
- $C = \frac{1}{1+x^2} \begin{bmatrix} 1 & x \\ -3x & 3 \end{bmatrix},$
- $D = \frac{4a^2}{1+x^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

Theorem 6. Equation (71) has no solutions for $|k| > 8$.

Proof. For functions $f, g : [-1, 1] \rightarrow \mathbb{C}^2$ let us define a standard scalar product:

$$(f|g) = \int_{-1}^1 \bar{f}^T g \, dx$$

Let us consider an operator $X := a \frac{d}{dx}$ and its hermitian conjugate $X^* = -\frac{d}{dx}a$. The eq. (71) takes the form:

$$X^* X v_k + \frac{k^2}{a^2} v_k + X^*(a B v_k) - i k C v_k - D v_k = 0$$

The left-hand side we denote by $L v_k$, where L is a linear operator and $v_k \in \ker L$, i.e. $L v_k = 0$. For $(v_k | L v_k)$ we have:

$$0 = (v_k | L v_k) = \|X v_k\|^2 + k^2 \left\| \frac{1}{a} v_k \right\|^2 + (X v_k | a B v_k) - i k \left(\frac{1}{a} v_k | a C v_k \right) - (v_k | D v_k).$$

Introducing real numbers $x := \frac{\|X v_k\|}{\|v_k\|}$, $y := \frac{\|\frac{1}{a} v_k\|}{\|v_k\|}$ we obtain:

$$x^2 \|v_k\|^2 + k^2 y^2 \|v_k\|^2 = -(X v_k | a B v_k) + i k \left(\frac{1}{a} v_k | a C v_k \right) + (v_k | D v_k),$$

and absolute value one can estimate as follows:

$$x^2 \|v_k\|^2 + k^2 y^2 \|v_k\|^2 \leq |(X v_k | a B v_k)| + |k| \left| \left(\frac{1}{a} v_k | a C v_k \right) \right| + |(v_k | D v_k)|.$$

From Cauchy-Schwarz inequality

$$x^2 \|v_k\|^2 + k^2 y^2 \|v_k\|^2 \leq x \|v_k\| \|a B v_k\| + |k| y \|v_k\| \|a C v_k\| + \|v_k\| \|D v_k\|,$$

and from $\|A v\| \leq \|A\| \|v\|$ we get:

$$x^2 \|v_k\|^2 + k^2 y^2 \|v_k\|^2 \leq (x \|a B\| + |k| y \|a C\| + \|D\|) \|v_k\|^2.$$

Hence

$$x^2 + k^2 y^2 \leq x \|a B\| + |k| y \|a C\| + \|D\|$$

or in an equivalent form:

$$\left(x - \frac{\|a B\|}{2} \right)^2 + \left(|k| y - \frac{\|a C\|}{2} \right)^2 \leq \|D\| + \frac{\|a B\|^2 + \|a C\|^2}{4}.$$

Positivity of $(x - \frac{\|a B\|}{2})^2$ gives

$$\left(|k| y - \frac{\|a C\|}{2} \right)^2 \leq \|D\| + \frac{\|a B\|^2 + \|a C\|^2}{4}$$

and

$$|k| \leq \frac{\|aC\| + \sqrt{4\|D\| + \|aB\|^2 + \|aC\|^2}}{2y}.$$

Definition of $y = \frac{\|\frac{1}{a}v_k\|}{\|v_k\|}$ and $a \leq \sqrt{2}$ gives $y \geq \frac{1}{\sqrt{2}}$, hence

$$|k| \leq \frac{\|aC\| + \sqrt{4\|D\| + \|aB\|^2 + \|aC\|^2}}{\sqrt{2}}.$$

A simple computation gives $\|D\| = 8$, $\|aB\| = \sqrt{6}$, $\|aC\| = 3\sqrt{2}$. Finally

$$|k| \leq \frac{3\sqrt{2} + \sqrt{32 + 6 + 18}}{\sqrt{2}} = 3 + \sqrt{28} \approx 8.29,$$

but k is integer hence $|k| \leq 8$. □

D Conformal vector field for extremal Kerr

We are looking for a vector field X in the following form:

$$X = A(x) \cos \phi \partial_x + B(x) \sin \phi \partial_\phi.$$

In coordinates (x, ϕ) the metric tensor $(g_{AB}) = \begin{pmatrix} m^2 \frac{1+x^2}{1-x^2} & 0 \\ 0 & 4m^2 \frac{1-x^2}{1+x^2} \end{pmatrix}$, hence

$$X_x = Am^2 \frac{1+x^2}{1-x^2} \cos \phi, \quad X_\phi = 4Bm^2 \frac{1-x^2}{1+x^2} \sin \phi.$$

The CVF equation

$$\nabla_A X_B + \nabla_B X_A = \nabla_C X^C g_{AB}$$

applied to our field X reduces to

$$\nabla_x X_x = \left(\frac{1+x^2}{1-x^2} A' + \frac{2x}{(1-x^2)^2} A \right) m^2 \cos \phi,$$

$$\nabla_\phi X_\phi = 4m^2 \left(\frac{1-x}{1+x^2} B - \frac{2x}{(1+x^2)^2} A \right) \cos \phi,$$

$$\nabla_\phi X_x + \nabla_x X_\phi = \left(4 \frac{1-x^2}{1+x^2} B' - \frac{1+x^2}{1-x^2} A \right) m^2 \sin \phi,$$

$$\nabla_C X^C = (A' + B) \cos \phi,$$

where $A' := \frac{dA}{dx}$. They can be written in an equivalent form:

$$\left(\frac{1+x^2}{1-x^2} A' + \frac{2x}{(1-x^2)^2} A \right) m^2 \cos \phi = \frac{1}{2} m^2 \frac{1+x^2}{1-x^2} (A' + B) \cos \phi,$$

$$4m^2 \left(\frac{1-x}{1+x^2} B - \frac{2x}{(1+x^2)^2} A \right) \cos \phi = 2m^2 \frac{1-x^2}{1+x^2} (A' + B) \cos \phi,$$

$$4 \frac{1-x^2}{1+x^2} B' - \frac{1+x^2}{1-x^2} A = 0,$$

and finally we obtain system of ODE's:

$$B = A' + \frac{4x}{1-x^4} A,$$

$$4B' \frac{1-x^2}{1+x^2} = A \frac{1+x^2}{1-x^2},$$

which leads to the second order ODE for the function B :

$$\left[\partial_x^2 - \frac{4x}{1-x^4} \partial_x - \frac{1}{4} \left(\frac{1+x^2}{1-x^2} \right)^2 \right] B(x) = 0$$

and $A(x) = 4 \left(\frac{1-x^2}{1+x^2} \right)^2 B'(x)$. We get the following solution:

$$B(x) = C_1 \cosh \left[\frac{1}{2} \left(x + \log \frac{1-x}{1+x} \right) \right] + C_2 \sinh \left[\frac{1}{2} \left(x + \log \frac{1-x}{1+x} \right) \right].$$

If we assume that the field X vanishes at one “pole” ($x = \pm 1$) we obtain the relation for constants C_i : $C_2 = \pm C_1$. For $C_2 = -C_1$ we have:

$$B(x) = C \sqrt{\frac{1+x}{1-x}} e^{-\frac{1}{2}x}, \quad A(x) = 2C \sqrt{\frac{1+x}{1-x}} \frac{1-x^2}{1+x^2} e^{-\frac{1}{2}x}.$$

Finally the CVF X takes the form:

$$X = C \sqrt{\frac{1+x}{1-x}} e^{-\frac{1}{2}x} \left(2 \frac{1-x^2}{1+x^2} \cos \phi \partial_x + \sin \phi \partial_\phi \right).$$

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